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LETTER TO THE EDITOR

Wilson expansions for an extended Potts model

R Oppermann†

Institut für Theoretische Physik der Universitat Frankfurt/Main, 6 Frankfurt 1, Robert-Mayer-Str 8, Germany

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Abstract. An n-component generalization of the continuous Potts model is studied both in the ordered and in the disordered phase by using Wilson's ϵ and $1/n$ expansions around the Heisenberg fixed point. The results indicate that the transition is always of first order for $d = 3$. For the case of a small first-order transition, we derive exponents in the critical region around the transition temperature and discuss the crossover to the isotropic behaviour. A close relation to equivalent results for a model with quartic anisotropy is also manifested.

Recent work on the three-state Potts (1952) model by Amit and Shcherbakov (AS) (1974) using the Callan–Symanzik equations for their ϵ expansion showed that the phase transition in this model should be first-order as predicted by Landau symmetry criteria even for arbitrarily small bare coupling constant u_3 of the interaction term with threefold symmetry. Since there are no fixed points displaying the symmetry of the Potts model, a continuous phase transition can only occur if the u_3 perturbation is irrelevant. This mechanism would be similar to the case of $SrTiO_3$ where the irrelevance of the quartic perturbation of cubic symmetry leads to a second-order transition (Aharony and Bruce 1974, Oppermann 1974). Estimating angle fluctuations around an ordering axis of the Potts model, Alexander (1974) predicted a continuous transition for small u_3. Hence we intend to study the influence of the corresponding fluctuations in $O(1/n)$ and $O(\epsilon)$ on the order of the transition and on the u_3 relevance using Wilson's diagram method (1972).

We start with the definition of the extended Potts model: it consists of $n/2$ Potts models which are coupled by isotropic four-spin interactions and is described by the Hamiltonian

$$\beta H = \frac{1}{2}\int_x \sum_{t=1}^{2}\sum_{\alpha=1}^{n/2} (r_0 s_{t\alpha}^2 + (\nabla s_{t\alpha})^2) + \frac{u_4}{4!}\int_x \left(\sum_{t\alpha} s_{t\alpha}^2\right)^2 - \frac{u_3}{3}\int_x \sum_\alpha (s_{1\alpha}^3 - 3s_{1\alpha}s_{2\alpha}^2) - \int_x \sum_{t\alpha} H_{t\alpha}s_{t\alpha}. \qquad (1)$$

We assume u_3 to be positive and small such that it can be used as an expansion parameter.

In order to obtain the large n limit of the equation of state (EQS), we assume the system to order along the s_{11} axis, where one of the absolute minima of the energy lies. Further we define $M = \langle s_{11}\rangle$, $H = H_{11}$, $H_{t\alpha} = 0$ for all transverse components and

† Present address: Institut für Theoretische Physik der Universität Heidelberg, D69 Heidelberg, Philosophenweg 19, Germany

observe that M^2, H^2, u_3^{-2} and u_4^{-1} are of O(n). Then taking into account that the free transverse propagators are given by $(q^2+u_3M)^{-1}$, we find for the EQS in O(1):

$$\frac{H}{M} = \tau + \frac{u_4}{6}M^2 - u_3 M - \frac{nu_4F}{6}\left(\frac{H}{M}+u_3M\right)^{d/2-1} \tag{2}$$

where $F = (2^d\pi^{d/2}\Gamma(d/2))^{-1}B(d/2-1, d/2-2)$, $d = 4-\epsilon$ is the space dimensionality, $2 < d \leqslant 4$, and $\tau = r_0 - r_{0c}$ with $r_{0c} = r_0(T_c)$. The EQS may be cast into scaling form. Here we solve the EQS close to or directly at the 'true' isotropic critical temperature T_c (ie $\tau = 0$) and study the Helmholtz energy $\beta A = \int H(M)\,dM$. Thus we obtain in the range between T_c and the temperature T_1 ($\gtrsim T_c$), where the first-order transition occurs, the following stable solution for the order parameter:

$$M = a(T)u_3^{(2-\epsilon)/(2+\epsilon)}, \qquad a(T_c) = (nF)^{2/(2+\epsilon)} \tag{3}$$

where $a(T)$ depends only slowly on the temperature. The stability limit T_2 of the metastable phase above T_1 is found as

$$T_2 - T_c = b\frac{2+\epsilon}{2-\epsilon}\left(\frac{2-\epsilon}{4}\right)^{4/(2+\epsilon)} u_3^{2(2-\epsilon)/(2+\epsilon)} \tag{4}$$

with $b = u_4(nF)^{4/(2+\epsilon)}/6$. At T_2 the order parameter is given by

$$M(T_2) = \left(\frac{2-\epsilon}{4}\right)^{2/(2+\epsilon)} M(T_c). \tag{5}$$

Now we turn to the study of the most interesting region of finite n and especially $n = 2$ (Potts model), looking first at O($1/n$) and O(ϵ) approximations of the transverse susceptibility by the Ward identity method (Wallace 1973). We arrive at the exact relation

$$0 = C_{11}^{\mathrm{T}}M + C_{\mathrm{T}}^{11}H/r_{\mathrm{T}} + u_3\int_x \sum_{t\alpha\beta}(C_{t\beta}^{1\alpha}\langle s_{t\beta}(x)(s_{1\alpha}^2(x) - s_{2\alpha}^2(x))s_{\mathrm{T}}(0)\rangle$$
$$-2C_{t\beta}^{2\alpha}\langle s_{t\beta}(x)s_{1\alpha}(x)s_{2\alpha}(x)s_{\mathrm{T}}(0)\rangle), \tag{6}$$

where C is an antisymmetric matrix which transforms the original spins by an infinitesimal rotation and the index T denotes an arbitrary transverse component. O(1) and O($1/n$) contributions to $r_{\mathrm{T}} = \chi_{\mathrm{T}}^{-1}(q=0)$ in addition to H/M arise only from the first term within the bracket of equation (6). We now compare equation (6) with the corresponding one obtained by Wallace for the case of quartic anisotropy Δ. We anticipate that, although the two equations look significantly different due to the different symmetry of the Δ and u_3 interactions, the u_3 symmetry effect to O($1/n, u_3$) results only in new binomial and counting factors, while the relevant parts of the integrals agree with those of the finite Δ case. Then we match u_3M and $(1/n)u_3M \ln M$ contributions to the power behaviour $r_{\mathrm{T}} \propto u_3M^{\psi_3}$ which yields

$$\psi_3 = 1 + 12A(\epsilon)/n + O(n^{-2}) \tag{7}$$

where $A(\epsilon) = \sin(\pi\epsilon/2)\Gamma(2-\epsilon)/(\pi\Gamma^2(2-\epsilon/2))$. In general one expects r_{T} to scale like $M^{\delta-1}$. This together with the definition $u_3 \propto M^{\phi_3/\beta}$ leads to the prediction of the scaling law

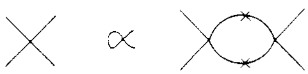
$$\psi_3 = (\delta-1)(1-\phi_3/\gamma). \tag{8}$$

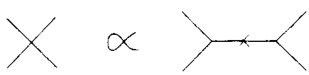
ϕ_3 is the crossover exponent of the u_3 perturbation. The exponents ψ_3 and ϕ_3 correspond to the similar defined exponents ψ_Δ and ϕ_Δ for the quartic perturbation which satisfy the same scaling law (Wallace 1973). Further we recognize that

$$\psi_3 = \psi_\Delta/2 + O(n^{-2}).$$

Alternatively ϕ_3 may also be calculated via the decomposition $\phi_3 = \gamma\alpha_3 - \gamma\alpha_4$, where α_3 describes scaling of the total three-point vertex $\Gamma_3 = \Gamma_{30} r^{\alpha_3}$ (Γ_{30} dimensionless), while α_4 is associated with the scaling of the factor u_3 of the vertex $\Gamma_3 \propto u_3 r^{\alpha_4}$ which yields the criterion for the order of the transition in agreement with that of the Callan–Symanzik theory. It states that for $u_3 \neq 0$ the isotropic fixed point with $\Gamma_{30}^* = 0$ is only approached with decreasing r if $\alpha_3 < \alpha_4$ (ϕ_3 negative), while for $\alpha_3 > \alpha_4$ the transition is of first order.

It is advantageous to calculate α_3 by the use of Polyakov's scaling theory (Polyakov 1969) based on the unitarity principle through which one also finds the scaling relation between α_3 and the exponent α_1 of the four-point vertex $\Gamma_4 \propto r^{\alpha_1}$:

 $\rightarrow \Gamma_4 \propto \Gamma_4^2 G^2 p^d \propto r^{(\epsilon - 2\eta)/(2-\eta)}$ (9a)

 $\rightarrow \Gamma_4 \propto \Gamma_3^2 G \rightarrow \Gamma_3 \propto r^{(1+\epsilon/2-3\eta/2)/(2-\eta)}$. (9b)

Thus we find with $\eta = O(\epsilon^2)$ and $\eta = [2\epsilon(2-\epsilon)/(4-\epsilon)]A(\epsilon)n^{-1} + O(n^{-2})$:

$$\alpha_3 = \begin{cases} (2+\epsilon)/4 + O(\epsilon^2) & (10a) \\ (2+\epsilon)/4 - \dfrac{\epsilon(2-\epsilon)A(\epsilon)}{4n} + O(n^{-2}) & (10b) \\ (1+\alpha_1)/2 \qquad \text{(exact)}. & (10c) \end{cases}$$

The exponent α_4 may be found in the ordered or the disordered phase by expanding u_{3p} which is defined by the symmetry-adjusted decomposition of the three-point vertex

$$\Gamma_3^{t\alpha,t'\alpha',t''\alpha''} = \frac{\int_{xy}\langle s_{t\alpha}(0)s_{t'\alpha'}(x)s_{t''\alpha''}(y)\rangle_c}{\int_x\langle s_{t\alpha}(0)s_{t\alpha}(x)\rangle \int_x\langle s_{t'\alpha'}(0)s_{t'\alpha'}(x)\rangle \int_x\langle s_{t''\alpha''}(0)s_{t''\alpha''}(x)\rangle}$$
$$= u_{34}(\delta_{t\alpha,\mathrm{L}}\delta_{t'\alpha',t''\alpha''} + \delta_{t'\alpha',\mathrm{L}}\delta_{t\alpha,t''\alpha''} + \delta_{t''\alpha'',\mathrm{L}}\delta_{t\alpha,t'\alpha'})$$
$$+ u_{3p}\delta_{\alpha\alpha'\alpha''}(\delta_{tt't''1} - (\delta_{t1}\delta_{t't''2} + \delta_{t'1}\delta_{tt''2} + \delta_{t''1}\delta_{tt'2})) \qquad (11)$$

where L means the longitudinal component (1, 1) and the δ are Kronecker deltas. u_{3p} is the interaction with Potts symmetry, while $u_{34} = -\frac{1}{3}u_4 M + $(higher orders) is a u_4-induced three-point interaction in the symmetry-broken phase and vanishes in the disordered phase. The result for α_4 is

$$\alpha_4 = (3/n)(2-\epsilon)A(\epsilon) + O(n^{-2}) = \alpha_2/2 + O(n^{-2}) \qquad (12)$$

where α_2 describes scaling of the factor of Δ of the vertex Γ_4 (see Ketley and Wallace 1974).

We use the $1/n$ expansion in order to estimate the critical spin dimension $\bar{n}_3$ where the change of the order of the transition occurs. As discussed above, $\bar{n}_3$ is defined by

$\alpha_3(\bar{n}_3)-\alpha_4(\bar{n}_3) = 0$. Combining equations (12) and (10*b*) we find

$$\bar{n}_3 = (12+\epsilon)\frac{2-\epsilon}{2+\epsilon}A(\epsilon). \tag{13}$$

This O(1/*n*) result states that for $d = 3$ isotropic second-order behaviour with $r \to 0$ (or $M \to 0$) is only approached if $n < \bar{n}_3(3) = 1{\cdot}8$. Since this O(1/*n*) result has to be considered as an upper limit, it indicates that in three dimensions the phase transition of the Potts model is always of first order. If one reformulates the u_3–Δ correspondence in terms of the crossover exponents, namely $2\phi_3 = \phi_\Delta+\gamma+O(n^{-2})$, one is already warned about the O(1/*n*) result for $\bar{n}_3$, since one expects $\gamma \geqslant 1$ and ϕ_Δ negative but small in magnitude for $n = 1, 2$ and 3.

For such small values of *n* one has to rely on the ϵ-expansion. We have to regard that M^{-2}, u_3^2 and u_4 are proportional to ϵ and, proceeding as before, obtain from expanding (6) and (11) to $O(\epsilon)$:

$$\psi_3 = 1+6\epsilon/(n+8) \tag{14}$$

$$\alpha_4 = 3\epsilon/(n+8) \to \alpha_3-\alpha_4 = \tfrac{1}{2}+\tfrac{1}{4}\epsilon(n-4)/(n+8). \tag{15}$$

It is interesting to note that the $O(\epsilon)$ approximation maintains the simple relations $\psi_3=\psi_\chi/2$ and $\alpha_4=\alpha_2/2$. By observation of the topology of higher-order graphs one expects that these relations hold in $O(\epsilon^2)$ too but break down in $O(\epsilon^3)$. The exponent $\alpha_3-\alpha_4$ agrees with the *n*-vector extension of the ($n = 2$) result of AS for $\partial\beta_3/\partial u_3$ at the isotropic fixed point. This quantity determines the stability of the isotropic fixed point against the u_3 perturbation. As can be seen, $\alpha_3-\alpha_4 > 0$, and therefore the transition is of first order for $d = 3$ and all $n \geqslant 0$. As above, one could try to set $\alpha_3-\alpha_4 = 0$ in order to find the critical spin dimension $\bar{n}_3$. However, the result $\bar{n}_3 = -8+6\epsilon+O(\epsilon^2)$ is not valid, since for $n \to -8$ the ϵ expansion breaks down, and equation (15) is no longer valid.

An encouraging agreement between our $O(\epsilon)$ and O(1/*n*) expansions and a numerical renormalization group study by Golner (1973) arises in the determination of the exponent ω_3^{-1}, which describes scaling of the order parameter discontinuity with u_3 according to $M^{\omega_3} \propto u_3$. For $d = 3$ and $n = 2$ we find 0·6 in $O(\epsilon)(\omega_3^{-1} = 1-\epsilon(n+2)/(n+8))$ and 0·69 in O(1/*n*), while Golner has obtained 0·56. One may also realize that $2\omega_3 = \omega_\Delta+\delta-1$.

A subsequent paper will contain, for example, expansions to higher orders and a discussion of u_3 corrections to the Heisenberg fixed point value of u_4.

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